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One-dimensional wave equations in disordered media

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Abstract. We prove that several one-dimensional wave equations for electrons, phonons and light propagation have all their states or proper modes exponentially localised when the medium is disordered. Vanishing of the DC conductivity is obtained for models describing electronic motion. The exact spectrum of these models is also obtained explicitly. Bounds on the localisation length are exhibited in some cases.

1. Introduction and statement of the results

Wave propagation in random media is an old problem, which regained vitality from the ideas of localisation theory as developed by Anderson, Mott and many others. In this direction, efforts have been mainly devoted to the Schrödinger equation with a random potential or its discrete analogue—the Anderson model—although many other wave equations are interesting for applications in condensed matter physics, astronomy, waveguide theory, radars and sonars, etc. In this paper we are going to study some of the most important such equations and give mathematical proofs of some properties of their normal modes.

Let n denote the points of the one-dimensional lattice $\mathbb{Z}$. The various equations that we are interested in are the stationary equations or proper modes equations associated with the following operators which act on the space of square integrable sequences $l^2(\mathbb{Z}) = \{\psi \mid \sum_{n\in\mathbb{Z}} |\psi(n)|^2 < \infty\}$:

$$(H_1\psi)(n) = -\psi(n+1) - \psi(n-1) + 2\psi(n) + V(n)\psi(n) \tag{1.1}$$

which corresponds to a discrete Schrödinger equation or an Anderson tight-binding model for evolution of electrons with diagonal disorder;

$$(H_2\psi)(n) = J(n, n+1)\psi(n+1) + J(n, n-1)\psi(n-1) \tag{1.2}$$

which corresponds to a tight-binding model for evolution of electrons with off-diagonal disorder;

$$(H_3\psi)(n) = J(n, n+1)\psi(n+1) + J(n, n-1)\psi(n-1) + V(n)\psi(n) \tag{1.3}$$

which is a tight-binding model for evolution of electrons with diagonal and off-diagonal disorder (operators H_1 and H_2 are of course special cases of H_3);

$$(H_4\psi)(n) = J(n, n+1)\psi(n+1) + J(n, n-1)\psi(n-1) - (J(n, n+1) + J(n, n-1))\psi(n) \tag{1.4}$$

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which corresponds to the evolution of phonons in harmonic crystals with random coupling forces;

$$(H_5\psi)(n) = (H_4\psi)(n) + V(n)\psi(n) \tag{1.5}$$

which does not seem to correspond to a specific problem, but which we shall need in our paper;

$$(H_6\psi)(n) = (1/m(n))(-\psi(n+1) - \psi(n-1) + 2\psi(n)) \tag{1.6}$$

which is the discrete Helmholtz equation associated with light propagation or also the equation corresponding to phonon evolution in harmonic crystals with random masses.

In all these equations we will always have $J(n, n+1) = J(n+1, n)$ and the $V(n)$, $J(n, n+1)$, $m(n)$, $n \in \mathbb{Z}$ will be independent random variables with distributions denoted r, p and s respectively. We will always suppose $\int |J| p(\mathrm{d}J) < \infty$. For equations H_1, H_3 and H_5, r will be supposed to be an absolutely continuous distribution, whose density, also denoted by r, will satisfy $\|r\|_\infty < \infty$ and $(1+|x|)r(x) \in L^2$. For equation H_1, in theorem 3, we will furthermore add to such a $V(n)$ an arbitrary given potential, for example a quasi-periodic one. For equations H_2 and H_4, p will be supposed to be absolutely continuous with density, also denoted p, satisfying $(1+J^2)p(J) < \infty$, and also in the case H_2, $p(J) = 0$ when $|J| < \varepsilon$. For equation H_6, the distribution s will be supposed absolutely continuous, with density s satisfying $(1 + 1/m^2)s(m) < \infty$.

Operators H_1 to H_5 are symmetric, and when all J and V are bounded uniformly with respect to n, they are bounded operators and hence self-adjoint. In the case of unbounded sequences of J or V, it is easy to check that they still define self-adjoint operators, although they are unbounded ones, and that the set of ψ with $\psi(n) = 0$ except for a finite number of n is a common core for them. The operator H_6, on the other hand, is not a symmetric operator as it stands; however, choosing new functions $\varphi(n) = (m(n))^{1/2}\psi(n)$, or in other words taking the new scalar product $(\psi_1, \psi_2) = \Sigma_{n\in\mathbb{Z}}\, m(n)\psi_1^*(n)\psi_2(n)$ for our Hilbert space, we obtain a self-adjoint operator.

The previous mathematical results concerning the normal modes of such operators, or in other words their spectral properties, are the following, all stated with probability one. First was proved the absence of an absolutely continuous spectrum for the operator H_6 on the half-line (Casher and Lebowitz 1971), and for H_1 on the whole line (Pastur 1974, 1980). The idea of the proof by Pastur together with the results on the Lyapunov exponents for random products of matrices (Guivarc'h 1981, Ledrappier and Royer 1980) allow us to obtain now the absence of an absolutely continuous spectrum for a large class of one-dimensional problems. Secondly, the proof has been obtained that a continuous Schrödinger equation with a random potential, or its discrete analogue H_1, has all its states exponentially localised (Golds'heid *et al* 1977, Molcanov 1978, Kunz and Souillard 1980, Carmona 1982, Royer 1982). The same results have been obtained for a large class of random second-order difference equations, including operators H_1 to H_6 by Lacroix (1981). Analogous results for the analogue of H_1 in a strip have been announced by Golds'heid (1980). Proof of an Anderson transition—namely a transition from pure point spectrum to absolutely continuous spectrum when the energy or the disorder is varied—for problem H_1 on a Bethe lattice has been announced (Kunz and Souillard 1981). Finally, concerning transport properties in one dimension, non-validity of Fourier's law for heat transfer has been proved for model H_6 (Casher and Lebowitz 1971) and vanishing of the electrical DC conductivity for model H_1 (Kunz and Souillard 1980).

In Kunz and Souillard (1980) a general formalism was developed to tackle such problems and we rely on it heavily in the present paper, where we introduce a variation on the proof of localisation achieved there. This variation is closely related to the independent work of Lacroix (1981), although we have not completely cleared up the links, and the conditions of applications are not exactly the same; it both makes possible a much simpler proof of localisation and allows us to extend it to a much larger class of systems (theorem 2) than the proof of Kunz and Souillard (1980). Furthermore, from this approach we can derive in the case of H_1 an upper bound on the localisation length and also prove that a Hamiltonian of type H_1 with a fixed potential, for example a quasi-periodic one, perturbed by a random potential again has all its states localised (theorem 3); these last results are, to our knowledge, the first in these directions.

We first mention the following theorem, giving the exact spectrum of the previous operators. We use the following notations: if A and B are two sets of real numbers, $A+B$ is the set of numbers $a+b$ for $a \in A$ and $b \in B$, $A \cdot B$ the set of numbers $a \cdot b$ for $a \in A$ and $b \in B$, and A^{-1} the set of numbers a^{-1} for $a \in A$. For the distributions r, p and s we let Supp denote the support of the distribution, i.e. for example Supp r is the set of V_0 such that $\text{Prob}\{V \in [V_0-\varepsilon, V_0+\varepsilon]\} > 0$ for any $\varepsilon > 0$.

Theorem 1. With probability one, the operators H_1 to H_6 have the following spectrum:

(1) $[0, 4] + \text{Supp}\, r$
(2) $[-2, 2] \cdot \text{Supp}\, p$
(3) $[-2, 2] \cdot \text{Supp}\, p + \text{Supp}\, r$
(4) $[0, 4] \cdot \text{Supp}\, p$
(5) $[0, 4] \cdot \text{Supp}\, p + \text{Supp}\, r$
(6) $[0, 4] \cdot (\text{Supp}\, s)^{-1}$.

Remarks. We shall not prove this result here. The case (1) has been proved in Kunz and Souillard (1980), and the other cases can be handled in the same spirit. These results hold even if the distributions of variables V, J, m are not absolutely continuous, for example are discrete distributions. They hold too for a very large class of stochastic processes, where the variables V, J, m are not independent from site to site. Analogous results hold for the corresponding equations in dimension d larger than 1; it is then sufficient to replace 2 by $2d$ and 4 by $4d$. For all these extensions we refer to the results and techniques of § III of Kunz and Souillard (1980).

We turn now to our results on the nature of the proper modes of equations (1)–(6) and to related results. The hypotheses on the distributions r, p and s have been stated above.

Theorem 2. The operators H_1 to H_6 have, with probability one, a pure point spectrum with exponentially decaying eigenfunctions, i.e. all their states are exponentially localised and satisfy for any bounded interval A (not containing 0 for cases 2, 4, 6)

$$\bar{\rho}(n, n'; A) = \left\langle \sum_{e \in S_p H \cap A} |\psi_e(n)\psi_e(n')| \right\rangle \leq C \exp\left(-\frac{|n-n'|}{\xi(A)}\right) \tag{1.7}$$

where the sum runs over all eigenvalues e of the operator in the interval A and ψ_e is the corresponding normalised eigenfunction, the brackets denoting the average over the disorder.

For operators H_1 to H_3 describing the motion of electrons, the static conductivity at zero temperature is null for any Fermi level E_F, except possibly for $E_F = 0$ in the case of H_2.

Theorem 3. For the operator H_1, the bound (1.7) is furthermore satisfied with $A = \mathbb{R}$ and the localisation length satisfies the uniform bound

$$\xi \leqslant \inf_{\substack{\eta > 0 \text{ such that} \\ 40\eta|\log \eta| < 1}} -2/\log\left(1 - \frac{\alpha(\eta)}{25}(1 - 40\eta|\log \eta|)^2\right) \tag{1.8}$$

where $\alpha(\eta) = 1 - \operatorname{Sup}_{|\omega| > \eta} |\tilde{r}(\omega)|$ and $\tilde{r}(\omega)$ is the Fourier transform of r.

In this case the static electric conductivity is zero for any temperature. All the previous results hold also if one adds to H_1 an arbitrary given potential.

Comments

(i) All these results also hold if the operators H_1 to H_6 are obtained as infinite volume limits of the analogous operators in finite boxes with boundary conditions such as free boundaries, periodic or antiperiodic boundaries, because such operators in finite boxes converge strongly in the resolvent sense toward the operators H_1 to H_6. They hold too for the operators on the semi-axis.

(ii) The localisation length diverges at $e = 0$ for the operators H_2, H_4 and H_6. For H_4 and H_6 this corresponds to the fact that $\psi(n) \equiv 1$ is a solution of $H\psi = e\psi$ for $e = 0$. In the case of H_2, it was sometimes claimed that $H_2\psi = e\psi$ possesses a localised state for $e = 0$ with an envelope decaying as $\exp(-\sqrt{n})$; but it is a simple consequence of the law of the iterated logarithm to check that the solution of $H_2\psi = 0$ is almost surely unbounded on both sides. In the three cases H_2, H_4 and H_6 one ought not to consider, however, that there exists an extended state at $e = 0$: extended states can have a physical meaning only if they form a continuum.

(iii) The operator H_4 is also the generator of a classical random walk in a one-dimensional random environment. It is known (Anshelevitch and Vologodskii 1981) that such a random walk has a usual diffusion behaviour with some appropriate diffusion constant, although the operator H_4 has all its proper modes localised. This phenomenon is crucially linked to the divergence of the localisation length at $e = 0$ in this case.

(iv) We remark that there exist problems with off-diagonal disorder which are equivalent to purely ordered systems: consider for example model H_2 on the half-line $n \geqslant 0$ with variables J taking only values $-\alpha$ and $+\alpha$ with respective probabilities p and $1 - p$. Consider then the eigenmode equation

$$J(n, n+1)\psi(n+1) + J(n, n-1)\psi(n-1) = e\psi(n),$$

and let

$$a(n+1) = \frac{J(n, n-1)}{J(n, n+1)} \frac{J(n-2, n-3)}{J(n-1, n-2)} \cdots$$

i.e. $a(n+1)$ is such that $a(n-1)J(n, n-1) = a(n+1)J((n, n+1)$. Hence, in the variables $\theta(n) = \psi(n)/a(n)$, the proper mode equation becomes $\theta(n-1) + \theta(n+1) = e\omega(n)\theta(n)$, where

$$\omega(n) = \frac{1}{J(n, n-1)^2} \frac{J(n-1, n-2)^2}{J(n-2, n-3)^2} \cdots$$

and so $\omega(n)$ is constant. Indeed such a model does not satisfy the hypothesis of our theorem 2.

(v) The upper bound (1.8) on the localisation length is to our knowledge the first one to be established. One should notice that this bound is a uniform one in the whole spectrum. One may wonder if this bound is a good one, and the answer is positive. Of course, the bound is not an exact estimation of the true localisation length: it is a strict upper bound to the length governing the decay of the function $\bar{\rho}(n, n')$ which could be taken as one definition of the localisation length. This latter length can be shown to be itself strictly larger than the inverse of the Lyapunov exponent which governs the rate of decay of the eigenfunctions (Carmona 1982), and which yields another definition of the localisation length. However, in the limit of small disorder, we have checked—at least for specific examples—that our upper bound (1.8) and the inverse of the Lyapunov exponent have the same scaling behaviour: this can be achieved, for example, in the Anderson model for which one has a rectangular distribution with width W for the diagonal disorder, and for which the Lyapunov exponent can be computed from the Herbert and Jones formula (1971), plotting in then a second-order approximation for small W of the density of states.

In § 2, we introduce some basic ideas for the proof of localisation, and we then give the framework of the proof of theorem 2. In § 3, we give the technical part of the proof of theorem 2. In § 4 we derive the results of theorem 3. The results on the electrical conductivity stated in theorems 2 and 3 follow respectively through theorems V.4 and V.2 of Kunz and Souillard (1980) together with the results of the present §§ 2–4.

2. An approach to the localisation problem

Following Kunz and Souillard (1980) (to be denoted by KS in the following) we denote by $\bar{\rho}_H(n, m; \mathrm{d}\lambda)$ for a given Hamiltonian H the absolute value of the spectral measure $E_H(\mathrm{d}\lambda)$ between the sites n and m, that is

$$\bar{\rho}_H(n, m; \mathrm{d}\lambda) = |\langle \delta_n, E_H(\mathrm{d}\lambda)\delta_m \rangle| \tag{2.1}$$

where δ_n is the function which takes the value 1 at the site n and 0 at the other sites. In this paper all our Hamiltonians depend on some disorder and $\bar{\rho}_\mu(n, m; \mathrm{d}\lambda)$ denotes the average of $\bar{\rho}_H(n, m; \mathrm{d}\lambda)$ with respect to the disorder. In the same way $\bar{\rho}_H^\Lambda$ and $\bar{\rho}_\mu^\Lambda$ denote the same quantities associated with the problem in a finite box Λ with given boundary conditions. In this case, if all the eigenvalues are non-degenerate $\bar{\rho}_H^\Lambda$ is given by

$$\bar{\rho}_H^\Lambda(n, m; A) = \sum_{\lambda \in A} |\psi_\lambda(n)\psi_\lambda(m)| \tag{2.2}$$

where ψ_λ is the normalised eigenfunction of the Hamiltonian H^Λ corresponding to the eigenvalue λ and where the sum runs over all the eigenvalues in A.

It is proved that if

$$\lim_{\Lambda \nearrow \mathbf{Z}} \bar{\rho}_\mu^\Lambda(n, m; A) \leq c(A) \exp[-\chi(A)|n-m|] \tag{2.3}$$

with $\chi(A)$ strictly positive, then with probability one (with respect to the disorder) the spectrum of H in A is pure point with the eigenfunctions exponentially localised.

Now we are going to give an empirical derivation of formula (2.11) below, for $\bar{\rho}_{\mu}^{\Lambda}$; we refer to KS, § VI for details and a rigorous derivation. We consider here the case where the disorder contains a random potential with absolutely continuous distribution (cases H_1, H_3, H_5), for which the change of variable (2.5) is judicious and leads to the formula (2.11) where $\bar{\rho}_{\mu}^{\Lambda}$ is expressed just as an integral. The other cases will be treated later in the same section. In order to unify the notations we shall denote by H_{nm} the element of matrices of the Hamiltonians between the sites n and m excluding the part due to the potential.

Let us consider a box $\Lambda = [-M, N]$ and the boundary conditions $\psi(-M-1) = \psi(N+1) = 0$ for the Hamiltonian restricted to Λ. Then following KS, the average of $\bar{\rho}_H^{\Lambda}$ over the diagonal disorder is by definition equal to

$$\langle \bar{\rho}_H^{\Lambda}(n, n'; A)\rangle_V = \int_{\mathbb{R}^{|\Lambda|}} \prod_{m\in\Lambda} r(V_m)\,\mathrm{d}V_m \sum_{\lambda\in A} |\psi_\lambda(n)\psi_\lambda(n')|. \tag{2.4}$$

We use the change of variables

$$\alpha_m = \psi(m)/\psi(0)$$

and

$$(\lambda, \alpha_m) \Leftrightarrow V_m = (\lambda\alpha_m - \Sigma_{m'} H_{mm'}\alpha_{m'})/\alpha_m. \tag{2.5}$$

Using the normalisation of ψ_λ, we get

$$|\psi(n)\psi(n')| = |\alpha_n\alpha_{n'}| \Big/ \sum_{m\in\Lambda} \alpha_m^2 \tag{2.6}$$

and (2.4) becomes

$$\langle \bar{\rho}_H^{\Lambda}(n, n'; A)\rangle = \int_A \mathrm{d}\lambda \int_{\mathbb{R}^{|\Lambda|-1}} \frac{|\alpha_n\alpha_{n'}|}{\Sigma\,\alpha_m^2} \prod_{m\in\Lambda} r\left(\frac{\lambda\alpha_m - \Sigma\, H_{mm'}\alpha_{m'}}{\alpha_m}\right) J(V; \lambda, \alpha) \prod_{m\in\Lambda\setminus\{0\}} \mathrm{d}\alpha_m \tag{2.7}$$

where $J(V; \lambda, \alpha)$ is the Jacobian of the change of variables. This Jacobian can be computed using theorem VI.1 of KS and the fact that in our cases the only off-diagonal terms of the Hamiltonians are $H_{mm\pm1}$, so

$$J(V; \lambda, \alpha) = \sum_{m\in\Lambda} \alpha_m^2 \prod_{m\in[-M-1,N]} |H_{mm+1}| \prod_{m\in\Lambda\setminus\{0\}} |\alpha_m|^{-1} |\alpha_{-M}\alpha_N|^{-1} \tag{2.8}$$

which gives us

$$\langle \bar{\rho}_H(n, n'; A)\rangle_V = \int \mathrm{d}\lambda \int |\alpha_n\alpha_{n'}||\alpha_{-M}\alpha_N|^{-1} \prod_{m\in[-M-1,N]} |H_{mm+1}|$$
$$\times \prod_{m\in\Lambda} \left[|\alpha_m^{-1}| r\left(\lambda - \sum_{m'} \frac{H_{mm'}\alpha_{m'}}{\alpha_m}\right)\right] \prod_{m\in\Lambda\setminus\{0\}} \mathrm{d}\alpha_m. \tag{2.9}$$

Now for the sake of simplicity, let us suppose $0 = n < n'$ and for all $m \neq 0$, let us set

$$y_m = \alpha_{m-1}/\alpha_m \qquad \text{if } 0 < m \leqslant N$$
$$y_m = \alpha_{m+1}/\alpha_m \qquad \text{if } -M \leqslant m < 0.$$

Clearly

$$J(y;\alpha)=\prod_{m\in\Lambda\setminus\{0\}}\frac{\partial y_m}{\partial\alpha_m}=\prod_{m>0}\left|\frac{\alpha_{m-1}}{\alpha_m^2}\right|\prod_{m<0}\left|\frac{\alpha_{m+1}}{\alpha_m^2}\right|$$

$$=\prod_{m\in\Lambda\setminus\{0\}}|\alpha_m|^{-1}|\alpha_{-M}\alpha_N|^{-1}. \qquad (2.10)$$

Then (2.9) becomes

$$\langle\bar\rho_H(0,n';A)\rangle_V=\int_A \mathrm{d}\lambda\int_{\mathbb{R}^{\Lambda-1}}|y_1y_2\ldots y_{n'}|^{-1}r_0(\lambda-H_{01}y_1^{-1}-H_{0-1}y_{-1}^{-1})$$

$$\times\prod_{m>0}r_m\left(\lambda-\frac{H_{mm+1}}{y_{m+1}}-H_{mm-1}y_m\right)\mathrm{d}y_m$$

$$\times\prod_{m<0}r_m\left(\lambda-H_{mm+1}y_m-\frac{H_{mm-1}}{y_{m-1}}\right)\mathrm{d}y_m\prod_{m\in[-M-1,N]}|H_{mm+1}| \qquad (2.11)$$

where $r_m(x)$ stands for $r(x-H_{mm})$.

As in the introduction, H_{mm+1} is now denoted by $-J_{mm+1}$. Thus H_{mm} equals 2 for the Hamiltonian H_1, 0 for H_3 and $J_{mm+1}+J_{mm-1}$ for H_5. We suppose that J_{mm+1} are independent random variables with identical distribution $\mathrm{d}\mu_J$. So, $\bar\rho_\mu^\Lambda$ is obtained by averaging $\langle\bar\rho_H^\Lambda\rangle_V$ over the remaining off-diagonal disorder, that is

$$\bar\rho_\mu^\Lambda(0,n';A)=\int\prod_{m\in[-M-1,N]}\mathrm{d}\mu_{J_{mm+1}}\langle\bar\rho_H^\Lambda(0,n';A\rangle_V. \qquad (2.12)$$

The case of the Hamiltonian H_1 may be considered as a special case of H_5 where the probability $\mathrm{d}\mu_J$ is a delta function at the value 1.

Our approach differs now from that of KS. We introduce the two operators T_0 and T_1 whose kernels with respect to the measure $|J|\,\mathrm{d}\mu_J\,\mathrm{d}x$ are given by

$$T_0(x',J'|x,J)=r(\lambda+J'x+Jx^{-1}) \qquad T_1(x',J'|x,J)=r(\lambda+J'x'+Jx^{-1})|x| \qquad (2.13)$$

for the Hamiltonian H_3 and

$$T_0(x'J'|xJ)=r(\lambda+J'x'+Jx^{-1}-J'-J) \qquad T_1(x'J'|xJ)=r(\lambda+J'x'+Jx^{-1}-J'-J)/|x| \qquad (2.14)$$

for the Hamiltonian H_5. In terms of these operators, a simple calculation gives us

$$\bar\rho_\mu^\Lambda(0,n';\lambda)=\int(T_1^{n'-1}T_0^{N-n'}F)(x,J)|x|^{-1}(T_0^MF)(x^{-1},J)|J|\,\mathrm{d}\mu_J\,\mathrm{d}x. \qquad (2.15)$$

where

$$F(x,J)=r(\lambda+Jx)\int|J'|\,\mathrm{d}\mu_{J'} \qquad \text{in the electronic case } (H_3)$$

and

$$F(x,J)=\int r(\lambda+Jx-J')|J'|\,\mathrm{d}\mu_{J'} \qquad \text{in the phononic case } (H_5). \qquad (2.16)$$

In this final expression for $\bar\rho_\mu^\Lambda$ the dependence on λ is contained in the definition of T_0, T_1 and F. Finally, let us remark that in all our cases T_0 and T_1 can be considered as operators acting only on one variable: for instance, in the 'electronic case' all the functions are of the form $F(x, J) = f(xJ)$, thus

$$(T_0F)(J, x) = \int r(\lambda + Jx + J'x'^{-1}) f(J'x')|J'| \, d\mu_{J'} \, dx' = g(Jx) \tag{2.17}$$

or equivalently

$$g(x) = \int r(\lambda + x + J^2 x'^{-1}) \, d\mu_J f(x') \, dx'. \tag{2.18}$$

In the same way

$$(T_1F)(J, x) = h(Jx) \tag{2.19}$$

where

$$h(x) = \int r(\lambda + x + J^2 x'^{-1})(|J|/|x'|) \, d\mu_J f(x') \, dx'. \tag{2.20}$$

Thus, from now on, in the case of H_3 we will use T_0 and T_1 as operators with kernels

$$T_0(x|x') = \int r(\lambda + x + J^2 x'^{-1}) \, d\mu_J \tag{2.21}$$

$$T_1(x|x') = \int r(\lambda + x + J^2 x'^{-1})(|J|/|x'|) \, d\mu_J. \tag{2.22}$$

With these new notations (2.15) becomes

$$\bar{\rho}_\mu^\Lambda(0, n'; \lambda) = \int (T_1^{n'-1} T_0^{N-n'} f)(x)|x|^{-1}(T_0^M f)(x^{-1}) \, dx \tag{2.23}$$

where $f(x)$ is now equal to $r(\lambda + x)\langle|J|\rangle$ where the brackets denote the average. In a similar way, for the phononic case all the functions $F(x, J)$ factorise as $f(J(x-1))$ and $\bar{\rho}_\mu^\Lambda$ satisfies again the equation (2.23) with T_0, T_1 and f given by

$$T_0(x|x') = \int r\left(\lambda + x + \frac{J^2}{x'+J} - J\right) d\mu_J \qquad T_1(x|x') = \int r\left(\lambda + x + \frac{J^2}{x'+J} - J\right) \frac{|J|}{|x'+J|} \, d\mu_J \tag{2.24}$$

$$f(x) = \int r(\lambda + x - J)|J| \, d\mu_J.$$

We have now the final expression for $\bar{\rho}_\mu^\Lambda$ in the case of H_1, H_3, H_5. Let us treat now the case of H_2, H_4 and H_6.

2.1. *Case of pure off-diagonal disorder: H_2 and H_4*

Until now we have only considered the case of the Hamiltonians H_1, H_3 and H_5. The reason is that we have presented a formal calculation to obtain the expression (2.23) for $\bar{\rho}_\mu^\Lambda$. This formal calculation is made rigorous by the results of KS, § VI in the case of H_1, H_3 and H_5, that is, if the Hamiltonian contains a random potential with an absolutely continuous distribution. However, we would like to use again the formula (2.23) in the case of Hamiltonians H_2, H_4 by taking in the definition of T_0, T_1 and f a delta function at zero instead of the function r.

Let us consider a Hamiltonian $H^{\Lambda}_{(J)}$ (of type H_2 or H_4) in a box Λ and a sequence of random potentials V^{Λ}_n; for instance, let us take potentials V^{Λ}_n independent on each site, with a density of probability r_n defined on a small interval around zero whose length goes to zero as n goes to infinity. Let us set

$$H^{\Lambda}_n(J) = H^{\Lambda}_{(J)} + V^{\Lambda}_n.$$

Then $H^{\Lambda}_n(J) \to H^{\Lambda}(J)$ in norm, and thus using the convergence of spectral measures and Fatou's lemma,

$$\bar{\rho}^{\Lambda}_{H(J)}(m, m'; A) \leqslant \varliminf_{n\to\infty} \langle \bar{\rho}^{\Lambda}_{H_n(J)}(m, m'; A)\rangle_{V_n} \tag{2.25}$$

where A is an open set and the angle brackets denote the average over the random potential. We can average (2.25) over the off-diagonal disorder to get, once more using Fatou's lemma,

$$\bar{\rho}^{\Lambda}_{\mu}(m, m'; A) \leqslant \varliminf_{n\to\infty} \langle \bar{\rho}^{\Lambda}_{H_n(J)}(m, m'; A)\rangle_{V_n,J}. \tag{2.26}$$

For any finite n, the average on the right-hand side is given in terms of the previous expression (2.23); now we would like to take the limit $n \to \infty$ inside the integral, that is to use directly the delta functions instead of the functions r_n. This can be done as follows as soon as the hypotheses of theorem 2 are satisfied and λ is different from 0.

In this situation $d\mu_J$ has to be absolutely continuous and we may assume at first that its density $P(J)$ is continuous. By a suitable limit procedure the results will remain valid for any $P(J)$, possibly not continuous, satisfying the hypotheses of theorem 2. Since the box Λ is finite, we may take the limit where the potential goes to zero site by site. Thus, beginning with the site N, we see that $f(r_n)$ goes to $f(\delta)$ which is a delta function in the worst case (H_2). The average $\langle |J| \rangle$ is always assumed to be finite and for simplicity will be set equal to one. Then the $f(r_n)$ have to be considered as the densities of a sequence of probabilities converging weakly to a probability formally defined by $f(\delta)$. Then since $P(J)$ is continuous, $T_0(x|x')$ is continuous with respect to x' and thus

$$(T_0(r)f(r_n))(x) \to (T_0(r)f(\delta))(x) \qquad \text{almost everywhere}$$

as n goes to infinity. Moreover

$$\int (T_0(r_n)f(\delta))(x)\,dx = \int (T_0(\delta)f(\delta))(x)\,dx = 1$$

and

$$(T_0(r_n)f(\delta))(x) \to (T_0(\delta)f(\delta))(x) \qquad \text{almost everywhere.}$$

Thus $T_0(r)f(\delta)$ is a sequence of convergent mass-preserving densities of probability and this ensures their convergence in L^1 toward $T_0(\delta)f(\delta)$. Repeating this argument site by site, we get the same result with T^k_0 instead of T_0. Later, in § 3, we will prove that $T^k_0(\delta)f(\delta)$ is uniformly in L^∞ as soon as $k \geqslant 2$ and $\lambda \neq 0$; as a matter of fact, it is easy to check that the same result holds if $T^k_0(\delta)$ is replaced by a product of k operators $T_0(\delta)$ associated with k different non-zero values of λ. This implies that $T^k_0(r_n)f(\delta)$ also satisfy uniform bounds when λ is non-zero and n large enough if one makes the following remarks: firstly $T_0(r_n)$ associated with the value parameter λ_0 is an average of a product of operators $T_0(\delta)$ associated with the parameter λ ranging in the interval $\lambda_0 + \text{support}\,\{r_n\}$, and secondly this interval does not contain zero when

n is large enough. These uniform bounds together with the L^1 convergence ensure the convergence in L^2 norm.

Now let us remark that the operator U defined by

$$(Uf)(x) = |x|^{-1}f(1/x) \tag{2.27}$$

is an isometry in L^2. Then using in (2.25) the convergence in L^2 for $T_0^M f$ and $T_0^{N-n'}f$, it is now sufficient to prove the norm convergence of $T_1(r_n)$ to $T_1(\delta)$ as n goes to infinity. But $\|T_1(\delta)\|_2$ is smaller than or equal to one by the Schwarz inequality and

$$T_1(r)f = r * T_1(\delta)f.$$

Thus

$$\|T_1(r_n) - T_1(\delta)\|_2 = \sup_{\|f\|=1} \|(T_1(r_n) - T_1(\delta))f\|_2 \leqslant \sup_{\|g\|=1} \|g - r_n * g\|_2$$

which goes to zero as r_n goes to a delta function. This ends the proof, showing that for H_2 and H_4 we can use (2.23) with r a delta function in order to get upper bounds of the kind (2.3) on the associated $\bar{\rho}_\mu$ functions.

2.2. *Phonons with random masses: H_6*

In order to solve the discrete Helmholtz problem H_6, we prove that this case is equivalent to the case of the operator H_4. Let us consider a phononic Hamiltonian H_4 in a finite box $[0, N]$ as before, except that we now set $J_{0-1} = J_{N,N+1} = 0$. The equations for an eigenvector φ associated with the eigenvalue e are

$$\begin{aligned} &-J_{01}(\psi_1 - \psi_0) = e\psi_0 \\ &-J_{nn+1}(\psi_{n+1} - \psi_n) - J_{nn-1}(\psi_{n-1} - \psi_n) = e\psi_n \qquad n \in [1, N-1] \\ &-J_{NN-1}(\psi_{N-1} - \psi_N) = e\psi_N. \end{aligned} \tag{2.28}$$

Let us set

$$\theta_n = J_{nn+1}(\psi_{n+1} - \psi_n) \qquad 0 \leqslant n \leqslant N-1.$$

Then these equations become for θ_n by subtraction

$$\begin{aligned} &-\theta_1 + 2\theta_0 = e\theta_0/J_{01} \\ &-\theta_{n+1} - \theta_{n-1} + 2\theta_n = e\theta_n/J_{nn+1} \qquad 1 \leqslant n \leqslant N-2 \\ &-\theta_{N-2} + 2\theta_{N-1} = e\theta_{N-1}/J_{NN-1}. \end{aligned} \tag{2.29}$$

Remark

Clearly, we have now only N equations and N variables θ_n $(0 \leqslant n \leqslant N-1)$, and this system is not equivalent to the previous one. The reason is that we have suppressed the solution ψ constant which is always an eigenvector associated with the eigenvalue 0 and corresponds to $\theta = 0$. Thus any solution of (2.28) for a non-zero energy corresponds to a solution of (2.29), and conversely given a solution θ_n, we have only one solution ψ orthogonal to the constant solution, that is a solution of

$$\sum \psi_n = 0 \qquad J_{nn+1}(\psi_{n+1} - \psi_n) = \theta_n.$$

Now the system (2.29) corresponds to the Hamiltonian H_6 with empty boundary condition. Moreover, the boundary conditions $J_{0-1} = J_{NN+1} = 0$ are not significant in the thermodynamic limit and the results obtained for H_4 may be used in the case of H_6. For instance, exponential decay for ψ yields at least the same exponential decay for θ. Thus all the localisation results obtained in the phononic case H_4 can be translated in the discrete Helmoltz case taking $1/m_n$ instead of J_{nn+1}.

We can turn now to the proof of the exponential localisation.

3. Proof of exponential localisation

Sketch of the proof

First, by the Schwarz inequality the equation (2.23) yields

$$\bar{\rho}_{\mu}^{\Lambda}(0, n; \lambda) \leq \|T_1^{n-1} T_0^{N-n} f\|_2 \|T_0^M f\|_2. \tag{3.1}$$

This inequality follows from the fact that the operator U defined by (2.27) is an isometry.

Now we see clearly that if we could prove that on the one hand $\|T_0^n f\|_2$ is uniformly bounded in n and that on the other hand the spectral radius of T_1 in $L^2(\mathrm{d}x)$ is strictly smaller than 1, then we would get

$$\varliminf_{\Lambda \nearrow \mathbf{Z}} \bar{\rho}_{\mu}^{\Lambda}(0, n; \lambda) \leq C(\lambda) \exp -[\chi(\lambda)|n|]. \tag{3.2}$$

Moreover, it is not difficult to prove that in our various cases $\chi(\lambda)$ and $C(\lambda)$ are continuous functions of λ (this technical point follows easily from the inequalities appearing in the proof below). Thus for any closed interval A (such that for all λ in A (3.2) holds) we conclude the existence of a $\chi(A)$ strictly positive such that

$$\varliminf_{\Lambda \nearrow \mathbf{Z}} \bar{\rho}_{\mu}^{\Lambda}(0, n; A) \leq C'(A) \exp -[\chi(A)|n|].$$

More precisely, the outline of the proof will be as follows.

(1) *Thermodynamic limit*

The functions f appearing in (2.23) have L^1 norm equal to $\langle |J| \rangle$. Since for all T_0

$$\int T_0(x|x')\,\mathrm{d}x = 1 \tag{3.3}$$

and T_0 positive, $\|T_0^n f\|_1$ is less than or equal to $\langle |J| \rangle$ by induction for all n. Hence it is now sufficient to find a uniform bound on $(T_0^n f)(x)$ to get a bound on $\|T_0^n f\|_2$ uniform in n. That will be the first technical point to prove.

(2) *Compactness of some power of T_1*

By the Schwarz inequality one can see that $\|T_1\|_2$ cannot be greater than one, so the spectral radius of T_1 can be at most equal to one. In order to prove that the radius is strictly smaller than one, the first step will be to show that T_1^2 is a compact operator. This point is necessary because T_1 is in fact of norm equal to one and thus otherwise it would be very difficult to conclude.

(3) *Spectral radius of T_1*

Now T_1^2 is compact so its spectrum is discrete and T_1 is of norm smaller than or equal to one. Let us suppose that the spectral radius of T_1 is one; then there would exist a function g in L^2 such that

$$T_1^2 g = g$$

and so

$$\|T_1 g\|_2 \leq \|g\|_2 = \|T_1^2 g\|_2 \leq \|T_1 g\|_2.$$

So we get

$$\|T_1 g\|_2 = \|g\|_2. \tag{3.4}$$

But we will show that this equality cannot be satisfied by any function g in L^2. This contradicts the hypothesis and thus the spectral radius of T_1 is strictly smaller than one.

3.1. *Thermodynamic limit*

First we have to take the thermodynamic limit. In our case this means showing that $\|T_0^n f\|_2$ is uniformly bounded over n. We do not give the proof in all cases, and we restrict ourselves to the electronic case with random potential (H_3) and the pure phononic case (H_4). The other cases are similar and we have stated the results in the introduction.

3.1.1. Electronic case (H_3). As we said previously, since $\|T_0^n f\|_1$ equals $\langle |J|\rangle$, it is sufficient to prove that $\|T_0^n f\|_\infty$ is uniformly bounded for n sufficiently large. The result is obvious as soon as $r(x)$ is bounded since

$$\|T_0^n f\|_\infty = \sup_x \left| \int r(\lambda + x + J^2 x'^{-1})\, d\mu_J (T_0^{n-1} f)(x')\, dx' \right| \leq \|r\|_\infty \|T_0^{n-1} f\|_1 = \|r\|_\infty \langle |J|\rangle. \tag{3.5}$$

3.1.2. Pure phononic case (H_4). The operators T_0 and T_1 have positive kernels and thus all the norms can be evaluated with positive functions. From now f will denote a positive function. Here $r(x)$ is not a function but a delta distribution and T_0 can be rewritten as

$$(T_0 f)(x) = \int d\mu_J f\left(-\frac{J(\lambda + x)}{\lambda + x - J}\right) \frac{J^2}{|\lambda + x - J|}. \tag{3.6}$$

Then

$$(T_0 f)(x) = \int P(J) \frac{J^2}{|\lambda + x|^2} f\left(-\frac{J(\lambda + x)}{\lambda + x - J}\right) d\left(\frac{J(\lambda + x)}{\lambda + x - J}\right) \tag{3.7}$$

$$\leq \frac{1}{|\lambda + x|^2} \sup J^2 P(J) \|f\|_1 \tag{3.8}$$

which gives a suitable bound for all x outside a neighbourhood of $-\lambda$. Now as x goes to $-\lambda$ using this preliminary result, we can get a bound for $T_0^2 f$:

$$(T_0^2 f)(x) = \int_{|J| \leq 2|\lambda + x|} P(J) \frac{J^2}{|\lambda + x|^2} (T_0 f)\left(-\frac{J(\lambda + x)}{\lambda + x - J}\right) d\left(\frac{J(\lambda + x)}{\lambda + x - J}\right)$$

$$+ \int_{|J| \geq 2|\lambda + x|} P(J) \frac{J^2}{|\lambda + x - J|^2} T_0 f\left(-\frac{J(\lambda + x)}{\lambda + x - J}\right) dJ \tag{3.9}$$

$$\leq 4\|P\|_\infty \|T_0 f\|_1 + \left(\int_{1 \geq |J| \geq 2|\lambda + x|} \|P\|_\infty \frac{J^2\, dJ}{|\lambda(\lambda + x - J) - J(\lambda + x)|^2} \right.$$

$$\left. + \int_{|J| \geq 1} P(J) \frac{J^2\, dJ}{|\lambda(\lambda + x - J) - J(\lambda + x)|^2} \right) \|f\|_1 \sup J^2 P(J). \tag{3.10}$$

Now the two remaining integrals are well defined and continuous with respect to $\lambda+x$ as $\lambda+x$ goes to zero, as soon as λ is not zero. It is easy to check that

$$\lim_{x\to-\lambda}|T_0^2 f(x)|\leq\left[4\|P\|_\infty+\sup_J J^2P(J)\left(\frac{2P_\infty}{\lambda^2}+\frac{1}{\lambda^2}\right)\right]\|f\|_1 \tag{3.11}$$

and thus $T_0^2 f$ is uniformly bounded.

Since $\|T_0^n f\|_1$ equals $\langle|J|\rangle$ for all n, this bound is valid for $T_0^n f$ as soon as n is larger than two.

Remark

In order to get (3.10) and (3.11), we have only used in (3.8) the decay at infinity of $T_0 f(x)$. Hence, if the second operator T_0 is associated with a parameter value λ different from that of the first one, the conclusion remains valid. This was necessary in § 2 in order to ensure the thermodynamic limit in the case of the pure off-diagonal disorder.

3.2. Compactness

As we said before, we have to prove that some power of the operator T_1 is compact (generally T_1 itself is not compact). In order to do so we use Riesz's criterion, that is, T is compact if:

(a) $Tf\to 0$ in the L^2 sense at infinity uniformly for f in the unit ball of L^2;
(b) $\lim_{h\to 0}\int(Tf(x+h)-Tf(x))^2\,dx=0$ uniformly for f in the unit ball.

3.2.1. Electronic case

$$|T_1 f(x)|=\left|\int r(\lambda+x+J^2x'^{-1})\frac{J}{|x'|}\,d\mu_J f(x')\,dx'\right| \tag{3.12}$$

$$\leq\int\left(\int r^2(\lambda+x+J^2x'^{-1})\frac{J^2}{|x'|^2}\,dx'\right)^{1/2}d\mu_J\|f\|_2$$

$$\leq\left(\int r^2(x)\,dx\right)^{1/2}\|f\|_2. \tag{3.13}$$

r is assumed to be in L^2; let us set $\|r\|_2=C$. Therefore using this bound and the Schwarz inequality, we get

$$|(T_1^2 f)(x)|\leq\int_{|J^2x'^{-1}|>|x|/2} r(\lambda+x+J^2x'^{-1})\frac{C}{|x'|}|J|\,d\mu_J\,dx'\|f\|_2$$

$$+\left(\int_{|J^2x'^{-1}|<|x|/2} r^2(\lambda+x+J^2x'^{-1})\frac{dx'J^2}{|x'|^2}\right)^{1/2}\|T_1 f\|_2 \tag{3.14}$$

$$\leq\int_{|x'|>|x|/2} r(\lambda+x+x')\frac{dx'}{|x'|}C\langle|J|\rangle+\left(\int_{|x'|<|x|/2} r^2(\lambda+x+x')\,dx'\right)^{1/2}. \tag{3.15}$$

In the last inequality we omit $\|f\|_2$ and $\|T_1 f\|_2$, since $\|f\|_2$ equals one and T_1 has a norm less than one. Now the first term on the right-hand side is bounded by $2C\langle|J|\rangle/|x|$ and the second term will have an equivalent bound if, say, $|x|r(x)$ belongs to L^2. Thus if $(1+|x|)r(x)$ belongs to L^2 and $\langle|J|\rangle$ is finite, $T_1^2 f$ goes to zero at infinity as $1/x$ and the property (a) is satisfied for T_1^2. The property (b) is satisfied if, for instance, (a) is satisfied and $(T_1^2 f)(x+h)$ goes to $(T_1^2 f)(x)$ uniformly in x and in f with norm one. This is easy to check since

$$|(T_1 f)(x+h)-(T_1 f)(x)|$$
$$=\left|\int [r(\lambda+x+h+J^2x'^{-1})-r(\lambda+x+J^2x'^{-1})]\frac{|J|}{|x'|}f(x')\,\mathrm{d}x'\,\mathrm{d}\mu_J\right|$$
$$\leq\left(\int (r(y+h)-r(y))^2\,\mathrm{d}y\right)^{1/2}. \tag{3.16}$$

This inequality is obtained by the Schwarz inequality, using that $\|f\|_2$ equals one. Moreover, this term goes to zero with h as soon as r belongs to L^2. Thus if $\langle|J|\rangle$ is finite and $(1+|x|)r(x)$ belongs to L^2, the operator T_1^2 is compact for all λ.

3.2.2. Pure phononic case (H₄)

(a) $T_1 f\to 0$ in the L^2 sense at infinity:

$$(T_1 f)(x)=\int \delta\left(\lambda+x+\frac{J^2}{x'+J}-J\right)\frac{|J|}{|x'+J|}P(J)\,\mathrm{d}J\,f(x')\,\mathrm{d}x'$$
$$=\int \frac{|J|}{|J-\lambda-x|}f\left(\frac{J(\lambda+x)}{J-\lambda-x}\right)P(J)\,\mathrm{d}J. \tag{3.17}$$

Thus

$$|(T_1 f)(x)|\leq\frac{1}{|\lambda+x|}\left[\int f^2\left(\frac{J(\lambda+x)}{J-\lambda-x}\right)\mathrm{d}\left(\frac{J(\lambda+x)}{J-\lambda-x}\right)\right]^{1/2}\left(\int J^2P^2(J)\,\mathrm{d}J\right)^{1/2} \tag{3.18}$$

and the result follows from the integrability of $J^2P^2(J)$ which is satisfied because we have supposed $JP(J)$ bounded and in L^1.

(b) Now we check the property (b) for the operator T_1^2 and since (a) holds, we have only to prove

$$\lim_{h\to 0}\int_{|x|<R}((T_1^2 f)(x+h)-(T_1^2 f)(x))^2\,\mathrm{d}x=0 \tag{3.19}$$

uniformly for f in the unit ball. This occurs for all non-zero λ since $(T_1^2 f)(x+h)$ goes to $(T_1^2 f)(x)$ uniformly in f and x outside a neighbourhood of $-\lambda$ and since $(T_1^2 f)(x)$ is uniformly bounded in this neighbourhood of $-\lambda$. Indeed,

$$|(T_1^2 f)(x)|\leq\|f\|_2\left(\int_{|J|<2|\lambda+x|}\frac{J^2}{|\lambda+x|^2}P^2(J)\,\mathrm{d}J\right)^{1/2}$$
$$+\|P\|_\infty\|f\|_2\int_{1>|J|>2|\lambda+x|}\frac{\|JP(J)\|_2|J|\,\mathrm{d}J}{|\lambda(J-\lambda-x)+J(\lambda+x)|}$$
$$+\|f\|_2\int_{|J|>1}\frac{\|JP(J)\|_2|J|P(J)\,\mathrm{d}J}{|\lambda(J-\lambda-x)+J(\lambda+x)|}. \tag{3.20}$$

Let us suppose $\lambda\neq 0$ and x goes to $-\lambda$; then this inequality gives us

$$(T_1^2 f)(x)\leq 4\|P\|_\infty+(\|JP(J)\|_2/\lambda)(2\|P\|_\infty+1) \tag{3.21}$$

which implies together with (3.8) that $(T_1^2 f)(x)$ is uniformly bounded. Now the continuity of $(T_1^2 f)(x)$ is obtained as follows:

$$(T_1^2 f)(x+h)-(T_1^2 f)(x)$$
$$=\int\left[\frac{|J|}{|J-\lambda-x-h|}(T_1 f)\left(\frac{J(\lambda+x+h)}{J-\lambda-x-h}\right)-\frac{|J|}{|J-\lambda-x|}\times(T_1 f)\left(\frac{J(\lambda+x)}{J-\lambda-x}\right)\right]P(J)\,\mathrm{d}J$$
$$=\int\frac{|J|}{|J-\lambda-x|}(T_1 f)\left(\frac{J(\lambda+x)}{J-\lambda-x}\right)\left(P(J')\frac{|\lambda+x|}{|\lambda+x+h|}\left|\frac{\mathrm{d}J'}{\mathrm{d}J}\right|-P(J)\right)\mathrm{d}J \tag{3.22}$$

where J' is the solution of the equation

$$\frac{J'(\lambda+x+h)}{J'-\lambda-x-h}=\frac{J(\lambda+x)}{J-\lambda-x}. \tag{3.23}$$

Then

$$|(T_1^2 f)(x+h)-(T_1^2 f)(x)|\leq\frac{1}{|\lambda+x|}\|T_1 f\|_2\left[\int\left(P(J')\frac{|\lambda+x|}{|\lambda+x+h|}\left|\frac{\mathrm{d}J'}{\mathrm{d}J}\right|-P(J)\right)^2 J^2\,\mathrm{d}J\right]^{1/2}. \tag{3.24}$$

It is easy to check that the last factor goes to zero with h as soon as $JP(J)$ is in L^2, and that $\lambda+x$ is different from zero. Thus $(T_1^2 f)(x+h)$ goes uniformly to $(T_1^2 f)(x)$ provided that $\lambda+x$ is not zero.

This ends the proof and tells us that T_1^2 is compact as long as λ is different from zero, $P(J)$ bounded and $JP(J)$ in L^2.

3.3. *The spectral radius*

3.3.1. Electronic case. Let us suppose that the spectral radius is one. Then as we said before, by compactness there exists a function f in L^2 such that

$$\|T_1 f\|_2=\|f\|_2$$

where

$$(T_1 f)(x)=\int r(\lambda+x+J^2x'^{-1})(|J|/|x'|)\,\mathrm{d}\mu_J f(x')\,\mathrm{d}x'$$
$$=\int r(\lambda+x+J^2x')(|J|/|x'|)f(x'^{-1})\,\mathrm{d}x'\,\mathrm{d}\mu_J. \tag{3.25}$$

Since $\||x|^{-1}f(1/x)\|_2$ is equal to $\|f\|_2$, there must exist a function g such that

$$\|g\|_2=\|T_1' g\|_2 \tag{3.26}$$

where

$$(T_1' g)(x)=\int r(\lambda+x+J^2x')|J|g(x')\,\mathrm{d}x'\,\mathrm{d}\mu_J \tag{3.27}$$

which becomes by Fourier transform

$$(\widetilde{T'_1 g})(k) = \tilde{r}(k) \int \tilde{g}(-kJ^2)\, e^{-ik\lambda}|J|\, d\mu_J. \tag{3.28}$$

By the Schwarz inequality

$$\left\| \int \tilde{g}(-kJ^2)\, e^{-ik\lambda}|J|\, d\mu_J \right\|_2 \leq \|\tilde{g}\|_2 = \|g\|_2. \tag{3.29}$$

Hence (2.26) cannot be satisfied since $|\tilde{r}(k)|$ is strictly smaller than one for all non-zero k. Thus if T_1^2 is compact its spectral radius is strictly smaller than one.

3.3.2. Pure phononic case. We proceed as in the previous case. But now T'_1 becomes

$$(T'_1 f)(x) = \int f\left(\frac{J-\lambda-x}{J(\lambda+x)}\right) \frac{1}{|\lambda+x|} P(J)\, dJ. \tag{3.30}$$

Then

$$\|T'_1 f\|_2 = \int f\left(\frac{J-\lambda-x}{J(\lambda+x)}\right) f\left(\frac{J'-\lambda-x}{J'(\lambda+x)}\right) \frac{dx}{|\lambda+x|^2} P(J)\, dJ P(J')\, dJ' \tag{3.31}$$

$$= \int f(x-1/J) f(x-1/J')\, dx\, P(J) P(J')\, dJ\, dJ'$$

$$\leq \|f\|_2. \tag{3.32}$$

The last inequality is obtained by the Schwarz inequality; hence the equality holds if and only if $f(x-1/J)$ is equal to $f(x-1/J')\, dx \otimes d\mu_J \otimes d\mu_J$ almost everywhere. Since $d\mu_J$ is absolutely continuous, $f(x)$ has to be constant almost everywhere and thus cannot be in L^2.

4. Localisation length

Let K_V and U denote the following operators acting on L^2 functions:

$$(K_V f)(x) = \int r(\lambda - V + x + x') f(x')\, dx' \qquad (Uf)(x) = |x|^{-1} f(1/x).$$

U is an isometry from L^2 to L^2. The operator T_1 of §§ 2, 3 associated with the case of the tight-binding Hamiltonian with diagonal disorder H_1 is nothing but UK_0. As a matter of fact, if one adds a given potential V to H_1, it is easy to check that a bound of type (3.1) remains valid with T_1^n replaced by $\Pi_{m=0}^{n-1} T_1(V(m))$ with $T_1(V(m)) = K_{V(m)}U$. The results of theorem 3 will be obtained as soon as we have proved that for all V and V'

$$\|K_V U K_{V'}\|^2 = \|K_0 U K_0\|^2 = \sup_{f, \int |f(x)|^2 dx = 1} |\langle f | K_0 U K_0 | f \rangle|$$

$$\leq 1 - \tfrac{1}{25}\alpha(\eta)(1 - 40\eta|\log \eta|)^2 \tag{4.1}$$

for all η such that $40\eta|\log\eta|<1$ and where

$$\alpha(\eta)=1-\sup_{|\omega|>\eta}|\tilde{r}(\omega)| \tag{4.2}$$

and $\tilde{r}(\omega)$ is the Fourier transform of r.

As a matter of fact, it will be easier to work in Fourier space, so we will consider normalised L^2 functions $\tilde{f}(\omega)$, $\|\tilde{f}\|=(\int|f(\omega)|^2\,\mathrm{d}\omega)^{1/2}=1$, and we will decompose $\tilde{f}$ as $\tilde{f}_1+\tilde{f}_2$ where $\tilde{f}_1$ (resp $\tilde{f}_2$) is zero for $|\omega|<\eta$ (resp $|\omega|>\eta$); f_1 and f_2 will be the Fourier transforms of $\tilde{f}_1$ and $\tilde{f}_2$ in x space. Then we have

$$\langle f|K_0UK_0|f\rangle=\langle f_1|K_0UK_0|f_1\rangle+2\langle f_1|K_0UK_0|f_2\rangle+\langle f_2|K_0UK_0|f_2\rangle \tag{4.3}$$

and the following inequalities will be proven below for $\eta<e^{-1}$:

$$|\langle f_1|K_0UK_0|f_1\rangle|\leqslant 40\eta|\ln\eta| \tag{4.4}$$

$$|\langle f_1|K_0UK_0|f_2\rangle|\leqslant\|f_2\|(1-\|f_2\|^2)^{1/2} \tag{4.5}$$

$$|\langle f_2|K_0UK_0|f_2\rangle|\leqslant\|f_2\|^2. \tag{4.6}$$

Admitting these inequalities, we obtain through (4.3), for $\eta<e^{-1}$,

$$|\langle f|K_0UK_0|f\rangle|\leqslant 40\eta|\ln\eta|+3\|f_2\|. \tag{4.7}$$

On the other hand, we see directly that

$$|\langle f|K_0UK_0|f\rangle|\leqslant 1-\alpha(\eta)\|f_2\|^2. \tag{4.8}$$

Then from (4.7) and (4.8) we can derive that for all η satisfying $40\eta|\ln\eta|<1$, we have

$$\begin{aligned}|\langle f|K_0UK_0|f\rangle|&\leqslant\sup_{f,\int|f(x)|^2\,\mathrm{d}x=1}\inf\{1-\alpha(\eta)\|f_2\|^2,\,40\eta|\ln\eta|+3\|f_2\|\}\\&\leqslant 1-\tfrac{1}{25}\alpha(\eta)(1-40\eta|\ln\eta|)^2.\end{aligned} \tag{4.9}$$

Hence the proof will be closed with the demonstration of the inequalities (4.4–4.6). Inequalities (4.5) and (4.6) are direct consequences of the fact that U is unitary, hence $|\langle g|K_0UK_0|h\rangle|\leqslant\|K_0g\|\|K_0h\|$ for all g and hL^2 functions, and that $\|K_0g\|\leqslant\|g\|$ as a consequence of the fact that $|\tilde{r}(\omega)|\leqslant 1$ for all ω.

So we are now left to prove (4.4), which is the important point. We have

$$|\langle f_1|K_0UK_0|f_1\rangle|=\left|\int\mathrm{d}\omega\,\mathrm{d}\omega'\,\tilde{f}(\omega)\bar{\tilde{r}}(\omega)G(\omega,\omega')\bar{\tilde{f}}(\omega')\tilde{r}(\omega')\right| \tag{4.10}$$

with

$$G(\omega,\omega')=\int|x|^{-1}\exp[\mathrm{i}(\omega'x^{-1}-\omega x)]\,\mathrm{d}x. \tag{4.11}$$

(G appears not to be well defined because of the $|x|^{-1}$ factor. However, one may restrict from the beginning the integrations to $\varepsilon\leqslant|x|\leqslant A$ and afterwards let $1/\varepsilon$ and A tend to infinity. The limits can be handled if one notes that all our norm estimations need only to be worked out with functions f such that $|fr|$ is integrable in addition to being square integrable.) Now $G(\omega,\omega')$ satisfies the following bound for $|\omega\omega'|<1$:

$$|G(\omega,\omega')|\leqslant -2\ln|\omega\omega'|+10. \tag{4.12}$$

This bound is derived in a straightforward way from (4.11) by considering variables

φ such as $e^{\varphi} = x(|\omega/\omega'|)^{1/2}$ and then displacing the line of integration of the variable φ on $\mathbb{R}$ of $\pm i\pi/2$ for φ positive or negative and according to the signs of ω and ω'.

From (4.12) and (4.10) and the fact that $|\tilde{r}(\omega)| \leq 1$, it follows that

$$|\langle f_1|K_0UK_0|f_1\rangle \leq 10\Big(\int |\tilde{f}_1(\omega)|\,d\omega\Big)^2 + 4\Big(\int |\tilde{f}_1(\omega)|\,d\omega\Big)\Big(\int |\tilde{f}_1(\omega)\ln\omega|\,d\omega\Big) \tag{4.13}$$

but through the Schwarz inequality we have

$$\int |\tilde{f}_1(\omega)|\,d\omega \leq \sqrt{2}\eta(1-\|f_2\|^2)^{1/2} \tag{4.14}$$

and

$$\int |\tilde{f}_1(\omega)\ln\omega\,d\omega \leq \sqrt{2}[\eta\ln^2\eta + 2\eta(1-\ln\eta)]^{1/2}(1-\|f_2\|^2)^{1/2}. \tag{4.15}$$

Using (4.14) and (4.15) into (4.13), we then obtain readily (4.4) for $\eta < e^{-1}$.

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